

Virasoro Symmetry of Constrained KP Hierarchies

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Abstract

Additional non-isospectral symmetries are formulated for the constrained Kadomtsev-Petviashvili (cKP) integrable hierarchies. The problem of compatibility of additional symmetries with the underlying constraints is solved explicitly for the Virasoro part of the additional symmetry through appropriate modification of the standard additional-symmetry flows for the general (unconstrained) KP hierarchy. We also discuss the special case of cKP – truncated KP hierarchies, obtained as Darboux-Bäcklund orbits of initial purely differential Lax operators. The latter give rise to Toda-lattice-like structures relevant for discrete (multi-)matrix models. Our construction establishes the condition for commutativity of the additional-symmetry flows with the discrete Darboux-Bäcklund transformations of cKP hierarchies leading to a new derivation of the string-equation constraint in matrix models.

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Introduction. Relations between integrable models and conformal symmetries have been studied intensely since the first early signs of their interconnection showed up in the literature in seventies [1]. More recently, the KdV hierarchy formulation of nonperturbative 2-d quantum gravity [2] in the framework of (multi-)matrix models prompted more studies in this field. The subsequent work pointed out the non-isospectral symmetry origin of the pertinent Virasoro constraints on the string partition function but remained mostly limited to the KdV-like reduction of the KP hierarchy since it was dealing with the double scaling limit of the matrix models [3].

Quite recently a new class of integrable systems appeared both in mathematical literature [4] and independently in physics literature [5], where the motivation came from Toda field theory and discrete matrix models. These systems belong to class of the so called constrained KP hierarchies (cKP) as they are obtained by a symmetry reduction (which generalizes the KdV type of reduction) from the underlying general (unconstrained) KP hierarchy. cKP hierarchies contain a large number of interesting hierarchies of soliton equations.

We address here the issue of formulation of additional non-isospectral Virasoro symmetry structure for the cKP hierarchies, or in different words, solving the problem for compatibility of the constraints with the additional non-isospectral symmetries of the original KP hierarchy. It is shown how initially the Virasoro algebra is broken by the constraints down to the $sl(2)$ subalgebra (containing Galilean and scaling symmetries) of the Virasoro algebra and how the Virasoro symmetry (for the Virasoro generators \mathcal{L}_n , $n \geq -1$) is recovered by modifying generators via adding a new structure consisting of ghost flows related to the plethora of (adjoint) eigenfunctions characteristic for the cKP hierarchies.

We also discuss the special case of cKP hierarchies – the so called truncated KP hierarchies obtained as Darboux-Bäcklund (DB) orbits of initial purely differential Lax operators. They are associated with Toda-lattice-like discrete integrable systems which can be naturally embedded in the cKP hierarchies and which are relevant for the description of discrete (multi-)matrix models. Applying here our construction establishes the condition for commutativity of additional-symmetry flows with the discrete Darboux-Bäcklund transformations. This condition sheds new light on the derivation of the string-equation constraint (string condition) for matrix models. Details of calculations will appear elsewhere [6].

Background on KP Hierarchy. We use the calculus of the pseudodifferential operators to describe the KP hierarchy of nonlinear evolution equations. In what follows the operator D is such that $[D, f] = f'$ with $f' = \partial f = \partial f / \partial x$ and it satisfies the generalized Leibniz rule (eq.(57) from Appendix).

The main object here is a pseudo-differential Lax operator L of a generalized KP hierarchy:

$$L = D^r + \sum_{j=0}^{r-2} v_j D^j + \sum_{i \geq 1} u_i D^{-i} \quad (1)$$

The associated Lax equations

$$\frac{\partial}{\partial t_l} L = \left[L_+^l, L \right] \quad l = 1, 2, \dots \quad (2)$$

(recall that $x \equiv t_1$) describe isospectral deformations of L . In (2) and below, the subscripts (\pm) of pseudo-differential operators indicate purely differential/pseudo-differential parts. Commutativity of the isospectral flows $\frac{\partial}{\partial t_l}$ (2) is then assured by the Zakharov-Shabat equations. One can also represent the Lax operator in terms of the dressing operator $W = 1 + \sum_1^\infty w_n D^{-n}$ through $L =$

$W D^r W^{-1}$. In this framework equation (2) is equivalent to the so called Wilson-Sato equation:

$$\frac{\partial}{\partial t_l} W = - \left(W D^l W^{-1} \right)_- W \quad (3)$$

For a given Lax operator L , which satisfies Sato's flow equation (2), we call the function Φ (Ψ), whose flows are given by the expression³:

$$\frac{\partial \Phi}{\partial t_l} = L_+^{\frac{l}{r}}(\Phi) \quad ; \quad \frac{\partial \Psi}{\partial t_l} = - (L^*)_+^{\frac{l}{r}}(\Psi) \quad l = 1, 2, \dots \quad (4)$$

an (*adjoint*) *eigenfunction* of L . In (4) we have introduced an operation of conjugation, defined by simple rules $D^* = -D$ and $(AB)^* = B^* A^*$. An eigenfunction, which in addition also satisfies the spectral equations $L\psi(\lambda, t) = \lambda\psi(\lambda, t)$ is called *Baker-Akhiezer (BA) function*.

Additional Symmetries for the KP hierarchy. The KP hierarchy has an infinite set of commuting symmetries associated with the isospectral flows described above (eq.(2)). However, the group of symmetries of the KP hierarchy is known to be much bigger. The extra symmetries are called “non-isospectral” or “additional” symmetries. A convenient approach to deal with symmetries of the integrable hierarchies of equations was developed by Orlov and Schulman (see [7, 8, 9]) and this is the approach we will use in this paper. Other important contributions to the subject of additional symmetries for the KP hierarchy were made by Fuchssteiner [10] and Chen et al. [11]. See also references [12] for the related discussion of the AKNS model, [13] for the truncated KP hierarchy and [14] for treatment of the generalized matrix hierarchies.

Let M be an operator “canonically conjugated” to L such that:

$$\left[L, M \right] = \mathbb{1} \quad , \quad \frac{\partial}{\partial t_l} M = \left[L_+^{\frac{l}{r}}, M \right] \quad (5)$$

The M -operator can be expressed in terms of dressing of the “bare” $M^{(0)}$ operator

$$M^{(0)} = \sum_{l \geq 1} \frac{l}{r} t_l D^{l-r} = X_{(r)} + \sum_{l \geq 1} \frac{l+r}{r} t_{r+l} D^l \quad ; \quad X_{(r)} \equiv \sum_{l=1}^r \frac{l}{r} t_l D^{l-r} \quad (6)$$

conjugated to the “bare” Lax operator $L^{(0)} = D^r$. The dressing gives

$$M = W M^{(0)} W^{-1} = W X_{(r)} W^{-1} + \sum_{l \geq 1} \frac{l+r}{r} t_{r+l} L^{\frac{l}{r}} = \sum_{l \geq 0} \frac{l+r}{r} t_{r+l} L_+^{\frac{l}{r}} + M_- \quad (7)$$

$$M_- = W X_{(r)} W^{-1} - t_r - \sum_{l \geq 1} \frac{l+r}{r} t_{r+l} \frac{\partial W}{\partial t_l} \cdot W^{-1} \quad (8)$$

where in (8) we used eqs.(3). Note that $X_{(r)}$ is a pseudo-differential operator satisfying $\left[D^r, X_{(r)} \right] = \mathbb{1}$.

The so called *additional (non-isospectral) symmetries* [7, 8] are defined as vector fields on the space of **KP** Lax operators (1) or, alternatively, on the dressing operator through their flows as follows:

$$\bar{\partial}_{k,n} L = - \left[\left(M^n L^k \right)_-, L \right] = \left[\left(M^n L^k \right)_+, L \right] + n M^{n-1} L^k \quad ; \quad \bar{\partial}_{k,n} W = - \left(M^n L^k \right)_- W \quad (9)$$

³For any (pseudo-)differential operator A and a function f , the symbol $A(f)$ will indicate application (action) of A on f as opposed to the symbol Af meaning just operator product of A with the zero-order (multiplication) operator f .

The additional flows commute with the usual KP hierarchy flows given in (2). But they do not commute among themselves, instead they form the $\mathbf{W}_{1+\infty}$ algebra (see e.g. [8]). One finds that the Lie algebra of operators $\bar{\partial}_{k,n}$ is isomorphic to the Lie algebra generated by $-z^n(\partial/\partial z)^k$. Especially for $n = 1$ this becomes an isomorphism to the Virasoro algebra $\bar{\partial}_{k,1} \sim -\mathcal{L}_{k-1}$, with $[\mathcal{L}_n, \mathcal{L}_k] = (n-k)\mathcal{L}_k$.

Constrained KP Hierarchy and Additional Symmetry. We now turn to the main problem of this letter, namely, compatibility of the additional Virasoro symmetry with the constraints defining the cKP hierarchy. We first introduce the symmetry constraints leading to the cKP hierarchy. Let ∂_{α_i} be vector fields, whose action on the standard KP Lax operator $Q = D + \sum_{i=0}^{\infty} u_i D^{-i-1}$ is induced by the (adjoint) eigenfunctions Φ_i, Ψ_i of Q through [4]:

$$\partial_{\alpha_i} Q \equiv [Q, \Phi_i D^{-1} \Psi_i] \quad (10)$$

We have the following proposition:

Proposition 1 *The vector fields ∂_{α_i} commute with the isospectral flows of the Lax operator Q :*

$$\left[\partial_{\alpha_i}, \frac{\partial}{\partial t_l} \right] Q = 0 \quad l = 1, 2, \dots \quad (11)$$

Definition 1 *The constrained KP hierarchy (denoted as $\text{cKP}_{r,m}$) is obtained by identifying the “ghost” flow $\sum_{i=1}^m \partial_{\alpha_i}$ with the isospectral flow ∂_r of the original KP hierarchy.*

Comparing (10) with equation (2) we find that for the Lax operator belonging to the $\text{cKP}_{r,m}$ hierarchy we have $Q_-^r = \sum_{i=1}^m \Phi_i D^{-1} \Psi_i$. Hence we are led to the Lax operator $L = Q^r$ given by

$$L = L_+ + \sum_{i=1}^m \Phi_i D^{-1} \Psi_i = D^r + \sum_{l=0}^{r-2} v_l D^l + \sum_{i=1}^m \Phi_i D^{-1} \Psi_i \quad (12)$$

and subject to the Lax equation (2). Therefore, we parametrize the $\text{cKP}_{r,m}$ hierarchy in terms of the Lax operator (12). Note that the (adjoint) eigenfunctions Φ_i, Ψ_i of the original Lax operator Q used in the above construction remain (adjoint) eigenfunctions for L (12) [15].

Applying the additional-symmetry flows (9) on L (12) for $n = 1$ we get

$$(\bar{\partial}_{k,1} L)_- = \left[(ML^k)_+, L \right]_- + (L^k)_- \quad (13)$$

Using the simple identities (59) and (61) from Appendix for the Lax operator (12), we are able to rewrite (13) as:

$$(\bar{\partial}_{k,1} L)_- = \sum_{i=1}^m (ML^k)_+ (\Phi_i) D^{-1} \Psi_i - \sum_{i=1}^m \Phi_i D^{-1} (ML^k)_+^* (\Psi_i) + \sum_{i=1}^m \sum_{j=0}^{k-1} L^{k-j-1} (\Phi_i) D^{-1} (L^*)^j (\Psi_i) \quad (14)$$

Here

$$L(\Phi_i) \equiv L_+(\Phi_i) + \sum_{j=1}^m \Phi_j \partial_x^{-1} (\Psi_j \Phi_i) \quad (15)$$

(and similarly for the adjoint counterpart) denotes action of L on Φ_i . Notice that $L^{k-j-1}(\Phi_i)$, $(L^*)^j(\Psi_i)$ are (adjoint) eigenfunctions of L (12). Hence, whereas the original L (12) belongs to the

class of $\text{cKP}_{r,m}$ hierarchies, the transformed Lax operator given by $\bar{\partial}_{k,1}L$ (cf. eq.(14)) belongs to a *different* class – $\text{cKP}_{r,m(k-1)}$ (for $k \geq 3$), since the number of eigenfunctions has increased.

For $k = 0, 1, 2$ the flow equations (14) can still be rewritten in the desired original $\text{cKP}_{r,m}$ form:

$$(\partial_\tau L)_- = \sum_{i=1}^m (\partial_\tau \Phi_i) D^{-1} \Psi_i + \Phi_i D^{-1} (\partial_\tau \Psi_i) \quad (16)$$

with $\partial_\tau \equiv \bar{\partial}_{k,1}$ ($k = 0, 1, 2$), where:

$$\bar{\partial}_{0,1} \Phi_i = (M)_+ (\Phi_i) \quad ; \quad \bar{\partial}_{0,1} \Psi_i = -(M)_+^* (\Psi_i) \quad (17)$$

$$\bar{\partial}_{1,1} \Phi_i = (ML)_+ (\Phi_i) + \alpha \Phi_i \quad ; \quad \bar{\partial}_{1,1} \Psi_i = -(ML)_+^* (\Psi_i) + \beta \Psi_i \quad \alpha + \beta = 1 \quad (18)$$

$$\bar{\partial}_{2,1} \Phi_i = (ML^2)_+ (\Phi_i) + L(\Phi_i) \quad ; \quad \bar{\partial}_{2,1} \Psi_i = -(ML^2)_+^* (\Psi_i) + L^*(\Psi_i) \quad (19)$$

Note an ambiguity on the right hand sides of (18).

The fact, that the action of the additional-symmetry flows can be defined on the (adjoint) eigenfunctions Φ_i, Ψ_i (as in (17)–(19)) in a way which is consistent with the additional-symmetry flows (13) on the constrained Lax operator (12), is equivalent to compatibility of the constraints with additional-symmetry flows for the KP hierarchy.

Since the additional flows satisfy an algebra $[\bar{\partial}_{l,1}, \bar{\partial}_{k,1}] = -(l-k) \bar{\partial}_{l+k-1,1}$ we have an isomorphism $\bar{\partial}_{k,1} \sim -\mathcal{L}_{k-1}$ with the Virasoro operators and equations (17)–(19) contain the $sl(2)$ subalgebra generators $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1$.

Since for $\partial_\tau \equiv \bar{\partial}_{k,1}$ $k \geq 3$, eq.(16) does not hold anymore due to absence of consistent definitions for $\bar{\partial}_{k,1} \Phi_i, \bar{\partial}_{k,1} \Psi_i$ generalizing (17)–(19) for higher k , it appears that the symmetry constraints behind the cKP hierarchies have broken the additional Virasoro symmetry down to the $sl(2)$ subalgebra.

To recover the complete Virasoro symmetry, our strategy will be to redefine the additional-symmetry generators. We first describe our technique for $k = 3$ in which case equation (13) contains a term:

$$(L^3)_- = \sum_{i=1}^m \Phi_i D^{-1} (L^*)^2 (\Psi_i) + \sum_{i=1}^m L(\Phi_i) D^{-1} L^* (\Psi_i) + \sum_{i=1}^m L^2(\Phi_i) D^{-1} \Psi_i \quad (20)$$

Note that the middle term in (20) is not of the form of required by equation (16). At this point we recall that for

$$X \equiv \sum_{k=1}^I M_k D^{-1} N_k \quad (21)$$

with definitions (12) and (21) we find using identity (60) from Appendix:

$$[X, L]_- = \sum_{k=1}^I \left(-L(M_k) D^{-1} N_k + M_k D^{-1} L^*(N_k) \right) + \sum_{i=1}^m \left(X(\Phi_i) D^{-1} \Psi_i - \Phi_i D^{-1} X^*(\Psi_i) \right) \quad (22)$$

According to 1 the flows generated by (22) will commute with the isospectral flows (2) provided M_i, N_i are (adjoint) eigenfunctions, which will be the case in what follows. Considering now as an example:

$$Y_3 \equiv \frac{1}{2} \sum_{i=1}^m \left(\Phi_i D^{-1} L^* (\Psi_i) - L(\Phi_i) D^{-1} \Psi_i \right) \quad (23)$$

we find that

$$\begin{aligned} [Y_3, L]_- &= -\left(L^3\right)_- + \frac{3}{2} \sum_{i=1}^m \left(\Phi_i D^{-1} (L^*)^2 (\Psi_i) + L^2 (\Phi_i) D^{-1} \Psi_i \right) \\ &+ \sum_{i=1}^m \left(Y_3 (\Phi_i) D^{-1} \Psi_i - \Phi_i D^{-1} Y_3^* (\Psi_i) \right) \end{aligned} \quad (24)$$

Hence $\left[-(ML^3)_- + Y_3, L \right]_-$ still has a form of (16). Therefore, it may be possible to find additional symmetries for the cKP model by combining the original $\bar{\partial}_{k,1}$ flows and the so-called ghost flows (associated with operators of Y_3 type). This will work provided that the above construction yields the Virasoro generator \mathcal{L}_2 obeying the correct algebra with the unbroken $sl(2)$ generators found above in (17)–(19). Note that each of the two terms in (23) could have been used with an appropriate factor to obtain the similar conclusion as in (24). The choice of the coefficients in Y_3 (23), apart from being the most symmetric combination, has the advantage that it will lead below to the correct Virasoro algebra commutators.

We now generalize the above manipulations to an arbitrary k . We introduce the pseudo-differential operators:

$$Y_k \equiv \sum_{i=1}^m \sum_{j=0}^{k-2} \left(j - \frac{1}{2}(k-2) \right) L^{k-2-j} (\Phi_i) D^{-1} (L^*)^j (\Psi_i) \quad ; \quad k \geq 2 \quad (25)$$

which are non-zero for $k \geq 3$. Notice that (22) and identity (61) from Appendix enable us to obtain:

$$\begin{aligned} [Y_k, L]_- &= \frac{k}{2} \sum_{i=1}^m \left(\Phi_i D^{-1} (L^*)^{k-1} (\Psi_i) + L^{k-1} (\Phi_i) D^{-1} \Psi_i \right) \\ &- \left(L^k \right)_- + \sum_{i=1}^m \left(-\Phi_i D^{-1} Y_k^* (\Psi_i) + Y_k (\Phi_i) D^{-1} \Psi_i \right) \end{aligned} \quad (26)$$

Our main result is contained in the following

Proposition 2 *The correct additional-symmetry flows for the cKP hierarchies (12), spanning the Virasoro algebra, are given by:*

$$\partial_k^* L \equiv \left[-\left(ML^k \right)_- + Y_k, L \right] \quad (27)$$

i.e., with the following isomorphism $\mathcal{L}_{k-1} \sim -\left(ML^k \right)_- + Y_k$, where Y_k are defined in (25).

Indeed, using (26) we first find that $(\partial_k^* L)_-$ can be cast in the form of (16) with

$$\partial_k^* \Phi_i = \left(ML^k \right)_+ (\Phi_i) + \frac{k}{2} L^{k-1} (\Phi_i) + Y_k (\Phi_i) \quad ; \quad \partial_k^* \Psi_i = -\left(ML^k \right)_+^* (\Psi_i) + \frac{k}{2} (L^*)^{k-1} (\Psi_i) - Y_k^* (\Psi_i) \quad (28)$$

Taking into account that $Y_i = 0$ for $i = 0, 1, 2$ we see that eq.(28) reproduces (17)–(19) (with ambiguity on the right hand side of (18) removed by fixing $\alpha = \beta = 1/2$). Hence $\partial_\ell^* = \bar{\partial}_{\ell,1}$ for $\ell = 0, 1, 2$.

Secondly, we note that the modified additional flows defined by (27) commute with the isospectral flows (2) and, due to (28), they preserve the form of the cKP Lax operator (12). The remaining question is whether they form a closed algebra. To answer this question we first calculate

$$\begin{aligned}\partial_\ell^* L^k(\Phi_i) &= \left(ML^\ell\right)_+ \left(L^k(\Phi_i)\right) + \left(k + \frac{1}{2}\ell\right) L^{k+\ell-1}(\Phi_i) \\ \partial_\ell^* (L^*)^k(\Psi_i) &= -\left(ML^\ell\right)_+^* \left((L^*)^k(\Psi_i)\right) + \left(k + \frac{1}{2}\ell\right) (L^*)^{k+\ell-1}(\Psi_i)\end{aligned}\quad (29)$$

valid for $\ell = 0, 1, 2$ and $k \geq 0$, and use it to obtain the following identity:

$$\bar{\partial}_{\ell,1} Y_k = \partial_\ell^* Y_k = \left[\left(ML^\ell\right)_+, Y_k \right]_- + (k - \ell) Y_{k+\ell-1} \quad ; \quad \ell = 0, 1, 2 \quad (30)$$

Accordingly, for $\ell = 0, 1, 2$ and any $k \geq 0$ we arrive at the fundamental commutation relations:

$$[\partial_\ell^*, \partial_k^*] L = (k - \ell) \partial_{k+\ell-1}^* L \quad (31)$$

The above discussion shows that $[\mathcal{L}_i, \mathcal{L}_k] = (i - k)\mathcal{L}_{i+k}$ for $i = -1, 0, 1$ ($sl(2)$ generators) and arbitrary k , where $\mathcal{L}_{k-1} \sim -\partial_k^*$. Accordingly, if \mathcal{L}_2 is associated with $Y_3 - (ML^3)_-$ all higher Virasoro operators can be obtained recursively from

$$\mathcal{L}_{n+1} = \frac{-1}{(n-1)} [\mathcal{L}_n, \mathcal{L}_1] \quad (32)$$

One can now easily show by induction that \mathcal{L}_k , $k \geq -1$ obtained in the above way form a closed Virasoro algebra up to terms, which commute with the $sl(2)$ generators. For illustration consider, e.g., $[\mathcal{L}_2, \mathcal{L}_3] \equiv Z$. Commuting \mathcal{L}_{-1} with Z we find $[\mathcal{L}_{-1}, Z] = 6\mathcal{L}_4$, which fixes Z to be $-\mathcal{L}_5$ up to terms commuting with the $sl(2)$ subalgebra. It is easy to see how to extend this argument to cover the whole Virasoro algebra.

Darboux-Bäcklund of cKP Hierarchies. Truncated KP Hierarchies. Let Φ be an eigenfunction of L defining a Darboux-Bäcklund transformation, i.e. :

$$\frac{\partial}{\partial t_l} \Phi = L_+^l(\Phi) \quad , \quad \tilde{L} = \left(\Phi D \Phi^{-1}\right) L \left(\Phi D^{-1} \Phi^{-1}\right) \quad , \quad \tilde{W} = \left(\Phi D \Phi^{-1}\right) W D^{-1} \quad (33)$$

Then the DB-transformed M operator (cf. (7)) acquires the form:

$$\tilde{M} = \left(\Phi D \Phi^{-1}\right) M \left(\Phi D^{-1} \Phi^{-1}\right) = \sum_{l \geq 0} \frac{l+r}{r} t_{r+l} \tilde{L}_+^l + \tilde{M}_- \quad (34)$$

$$\tilde{M}_- = \tilde{W} \tilde{X}_{(r)} \tilde{W}^{-1} - t_r - \sum_{l \geq 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} \tilde{W} \cdot \tilde{W}^{-1} \quad (35)$$

where $\tilde{X}_{(r)} = D X_{(r)} D^{-1}$ with $X_{(r)}$ as in (6). Clearly $\tilde{X}_{(r)}$, like $X_{(r)}$, is also admissible as canonically conjugated to D^r .

In particular, for L belonging to a cKP hierarchy (12) we consider special class of DB transformations (33) which preserve the constrained cKP form of L :

$$\tilde{L} = T_a L T_a^{-1} = \tilde{L}_+ + \sum_{i=1}^m \tilde{\Phi}_i D^{-1} \tilde{\Psi}_i \quad , \quad T_a \equiv \Phi_a D \Phi_a^{-1} \quad (36)$$

$$\tilde{\Phi}_a = T_a L(\Phi_a) \quad , \quad \tilde{\Psi}_a = \Phi_a^{-1} \quad (37)$$

$$\tilde{\Phi}_i = T_a(\Phi_i) \quad , \quad \tilde{\Psi}_i = T_a^{-1*} \Psi_i = -\Phi_a^{-1} \partial_x^{-1} (\Psi_i \Phi_a) \quad , \quad i \neq a \quad (38)$$

where the DB-generating $\Phi \equiv \Phi_a$ coincides with one of the eigenfunctions of the initial L (12).

Let us consider the following generic class of DB orbits on $\text{cKP}_{r,m}$: within each subset of m successive DB steps we perform DB transformations (36) w.r.t. the m different eigenfunctions of (12). Repeated use of a composition formula for Wronskians (see eqs.(66)–(68) from Appendix) leads us to the following explicit expressions for the eigenfunctions and the τ -function after $km + l$ steps ($1 \leq l \leq m$) of successive DB transformations (see also ref.[16]) :

$$\begin{aligned} \Phi_i^{(km+l)} &= T_l^{(km+l-1)} \dots T_1^{(km)} T_m^{(km-1)} \dots T_1^{((k-1)m)} \dots T_m^{(m-1)} \dots T_1^{(0)} \chi_i^{(k\pm)} \\ &= \frac{W \left[\Phi_1^{(0)}, \dots, \Phi_m^{(0)}, \chi_1^{(1)}, \dots, \chi_m^{(1)}, \dots, \chi_1^{(k-1)}, \dots, \chi_m^{(k-1)}, \chi_1^{(k)}, \dots, \chi_l^{(k)}, \chi_i^{(k\pm)} \right]}{W \left[\Phi_1^{(0)}, \dots, \Phi_m^{(0)}, \chi_1^{(1)}, \dots, \chi_m^{(1)}, \dots, \chi_1^{(k-1)}, \dots, \chi_m^{(k-1)}, \chi_1^{(k)}, \dots, \chi_l^{(k)} \right]} \end{aligned} \quad (39)$$

$$\begin{aligned} \chi_i^{(k+)} &\equiv \chi_i^{(k+1)} \quad \text{for } 1 \leq i \leq l \quad ; \quad \chi_i^{(k-)} \equiv \chi_i^{(k)} \quad \text{for } l+1 \leq i \leq m \\ \frac{\tau^{(km+l)}}{\tau^{(0)}} &= \Phi_l^{(km+l-1)} \dots \Phi_1^{(km)} \Phi_m^{(km-1)} \dots \Phi_1^{((k-1)m)} \dots \Phi_m^{(m-1)} \dots \Phi_1^{(0)} \\ &= W \left[\Phi_1^{(0)}, \dots, \Phi_m^{(0)}, \chi_1^{(1)}, \dots, \chi_m^{(1)}, \dots, \chi_1^{(k-1)}, \dots, \chi_m^{(k-1)}, \chi_1^{(k)}, \dots, \chi_l^{(k)} \right] \end{aligned} \quad (40)$$

where the upper indices in brackets indicate the order of the corresponding DB step, the zero index referring to the “initial” cKP Lax operator, and where we have employed the short-hand notations:

$$T_i^{(k)} \equiv \Phi_i^{(k)} D \left(\Phi_i^{(k)} \right)^{-1} \quad ; \quad \chi_i^{(s)} \equiv \left(L^{(0)} \right)^s \left(\Phi_i^{(0)} \right) \quad , \quad i = 1, \dots, m \quad (41)$$

As seen from (36)–(38) and (39), the DB orbit $L^{(k)} = \left(L^{(k)} \right)_+ + \sum_{i=1}^m \Phi_i^{(k)} D^{-1} \Psi_i^{(k)}$ of $\text{cKP}_{r,m}$, starting from a purely differential initial $L^{(0)} = \left(L^{(0)} \right)_+$, defines a class of *truncated* $\text{cKP}_{r,m}$ hierarchies where the m adjoint eigenfunctions $\Psi_i \equiv \Psi_i^{(k)}$ are not independent from the m eigenfunctions $\Phi_i \equiv \Phi_i^{(k)}$ since both are parametrized in terms of m independent functions $\Phi_i^{(0)}$ only.

As a simple example of truncated cKP hierarchies, consider formulas (39)–(40) for the DB orbit of $\text{cKP}_{1,m}$ hierarchy (the so called “multi-boson” reduction of the general KP hierarchy) starting from a “free” initial $L^{(0)} = D$. In this case we have to substitute in (39)–(40) :

$$\chi_i^{(s)} \equiv \partial^s \Phi_i^{(0)} \quad , \quad \Phi_i^{(0)} = \int_{\Gamma} d\lambda \phi_i^{(0)}(\lambda) \exp \left\{ \sum_{r \geq 1} \lambda^r t_r \right\} \quad (42)$$

with arbitrary “densities” $\phi_i^{(0)}(\lambda)$ (and with appropriate contour Γ such that the λ -integrals exist). A special feature of truncated $\text{cKP}_{1,m}$ is that their dressing operators are truncated (having only finite number of terms in the pseudo-differential expansion, cf. [13]) :

$$W^{(km+l)} = T_l^{(km+l-1)} \dots T_1^{(km)} T_m^{(km-1)} \dots T_1^{((k-1)m)} \dots T_m^{(m-1)} \dots T_1^{(0)} D^{-km-l} = \sum_{j=0}^{km+l} w_j^{(km+l)} D^{-j} \quad (43)$$

where notations (41) were used.

The particular case $m = 1$ of (36)–(38), (39)–(40) yields:

$$L^{(k+1)} = \left(\Phi^{(k)} D \Phi^{(k)} \right)^{-1} L^{(k)} \left(\Phi^{(k)} D^{-1} \Phi^{(k)} \right) = D + \Phi^{(k+1)} D^{-1} \Psi^{(k+1)} \quad (44)$$

$$\Phi^{(k+1)} = \Phi^{(k)} \left(\ln \Phi^{(k)} \right)'' + \left(\Phi^{(k)} \right)^2 \Psi^{(k)} \quad , \quad \Psi^{(k+1)} = \left(\Phi^{(k)} \right)^{-1} \quad (45)$$

$$\Phi^{(n)} = \frac{W_{n+1}[\phi, \partial\phi, \dots, \partial^n \phi]}{W_n[\phi, \partial\phi, \dots, \partial^{n-1} \phi]} \quad , \quad \tau^{(n)} = W_n[\phi, \partial\phi, \dots, \partial^{n-1} \phi] \quad (46)$$

where

$$\phi \equiv \Phi^{(0)} = \int d\lambda \phi^{(0)}(\lambda) \exp \left\{ \sum_{r=1}^{\infty} t_r \lambda^r \right\} \quad (47)$$

The hierarchies given by (44) are generalizations of the Burgers-Hopf hierarchy defined by $L^{(1)} = D + \phi(\ln \phi)'' D^{-1} \phi^{-1}$.

Additional Symmetries versus DB Transformations for cKP Hierarchies. String Condition.

With the help of identities (62)–(65) from Appendix we find the following explicit form of the DB transformation of the operators Y_k (25) :

$$T_a Y_k T_a^{-1} = \tilde{Y}_k - \left(\tilde{L}^{(a)} \right)_-^{k-1} + \left\{ T_a \left(Y_k + \frac{k}{2} L^{k-1} \right) (\Phi_a) \right\} D^{-1} \Phi_a^{-1} \quad (48)$$

$$\left(\tilde{L}^{(a)} \right)_-^{k-1} \equiv \sum_{j=0}^{k-2} \tilde{L}^{k-j-2}(\Phi_a) D^{-1} \left(\tilde{L}^* \right)^j (\Psi_a) \quad (49)$$

Here \tilde{L} , T_a are as in (36) and the DB-transformed \tilde{Y}_k have the same form as Y_k in (25) with all (adjoint) eigenfunctions substituted with their DB-transformed counterparts as in (36)–(38). Also notice that in the particular case of cKP $_{r,1}$ hierarchies $\left(\tilde{L}^{(a)} \right)_-^{k-1}$ (49) coincides with the (pseudo-differential part of the power of the) full cKP $_{r,1}$ Lax operator (cf. eq.(12) for $m = 1$ and (61)).

Taking into account (36)–(38) and (48)–(49) we arrive at the following important

Proposition 3 *The additional-symmetry flows (27) for cKP $_{r,1}$ hierarchies (eq.(12) with $m = 1$) commute with the Darboux-Bäcklund transformations (36) preserving the form of cKP $_{r,1}$, up to shifting of (27) by ordinary isospectral flows. Explicitly we have:*

$$\partial_k^* \tilde{L} = - \left[\left(\tilde{M} \tilde{L}^k \right)_- - \tilde{Y}_k, \tilde{L} \right] + \frac{\partial \tilde{L}}{\partial t_{k-1}} \quad (50)$$

Proposition 3 shows that the additional-symmetry flows (27) are well-defined for all cKP $_{r,1}$ Lax operators belonging to a given DB orbit of successive DB transformations. Notice that it is precisely the class of (truncated) cKP $_{r,1}$ hierarchies which is relevant for the description of discrete (multi-)matrix models [17, 15, 16].

Motivated by applications to (multi-)matrix models (see ref.[6]), one can require invariance of cKP hierarchies under some of the additional-symmetry flows, e.g., under the lowest one $\partial_0^* \equiv \bar{\partial}_{0,1}$ known as “string-equation” constraint (string condition) in the context of the (multi-)matrix models:

$$\partial_0^* L = 0 \quad \rightarrow \quad \left[M_+, L \right] = -\mathbb{1} \quad ; \quad \partial_0^* \Phi = 0 \quad \rightarrow \quad M_+ \Phi = 0 \quad (51)$$

Eqs.(51), using second eq.(5),(7) and first eq.(28) for $k = 0$, lead to the following constraints on L (12) and its DB-generating eigenfunction Φ , respectively:

$$\sum_{l \geq 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} L + \left[t_1, L \right] \delta_{r,1} = -\mathbb{1} \quad (52)$$

$$\left(\sum_{l \geq 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} + t_r \right) \Phi = 0 \quad (53)$$

Recall now the formula (40) for the τ -function of the $\text{cKP}_{r,m}$ hierarchy (12). Noticing that the eigenfunctions $\Phi^{(k)}$ of the DB-transformed Lax operators $L^{(k)}$ satisfy the *same* constraint eq.(53) irrespective of the DB-step k , we arrive at the following result (“string-equation” constraint on the τ -functions) :

Proposition 4 *The Wronskian τ -functions (40) of $\text{cKP}_{r,m}$ hierarchies (12), invariant under the lowest additional symmetry flow (51), satisfy the constraint equation:*

$$\left(\sum_{l \geq 1} \frac{l+r}{r} t_{r+l} \frac{\partial}{\partial t_l} + n t_r \right) \frac{\tau^{(n)}}{\tau^{(0)}} = 0 \quad (54)$$

As the simplest illustration of this proposition, consider the discrete one-matrix model corresponding to the generalized Burgers-Hopf hierarchy, *i.e.*, to the chain of the Lax operators connected via DB transformations as described in eqs.(44)–(45), but with the *additional restriction* on $\phi \equiv \Phi^{(0)}$ (47) (coming from the orthogonal polynomial formalism) :

$$\phi = \int d\lambda \exp \left(\sum_{r=1}^{\infty} t_r \lambda^r \right) \quad , \quad \text{i.e.} \quad \phi^{(0)}(\lambda) = 1 \quad (55)$$

The initial “free” eigenfunction (55) obeys the constraint eq.(53) (for $r = 1$) and, therefore, proposition 4 (with $r = 1$) yields precisely the “string-equation” in the one-matrix model:

$$\mathcal{L}_{-1}^{(N)} W_N[\phi, \partial\phi, \dots, \partial^{N-1}\phi] = 0 \quad , \quad \mathcal{L}_{-1}^{(N)} \equiv \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} + N t_1 \quad (56)$$

Furthermore, as one can check directly [6], the Wronskian τ -function (second eq.(46)) with ϕ restricted as in (55) automatically satisfies all higher Virasoro constraints. Thus, we conclude that for the particular class of cKP hierarchies – the generalized Burgers-Hopf hierarchies (44)–(47), invariance under the lowest additional-symmetry flow automatically triggers invariance under all higher additional-symmetry flows as well.

Appendix: Technical Identities. We list here for convenience a number of useful technical identities, which have been used extensively throughout the text.

We work with calculus of pseudo-differential operators based on the generalized Leibniz rule:

$$D^n f = \sum_{j=0}^{\infty} \binom{n}{j} (\partial^j f) D^{n-j} \quad (57)$$

For an arbitrary pseudo-differential operator A we have the following identity:

$$\left(\chi D \chi^{-1} A \chi D^{-1} \chi^{-1} \right)_+ = \chi D \chi^{-1} A_+ \chi D^{-1} \chi^{-1} - \chi \partial_x \left(\chi^{-1} A_+(\chi) \right) D^{-1} \chi^{-1} \quad (58)$$

where A_+ is the differential part of $A = A_+ + A_- = \sum_{i=0}^{\infty} A_i D^i + \sum_{i=-\infty}^{-1} A_i D^i$.

For a purely differential operator K and arbitrary functions f, g we have an identity

$$[K, f D^{-1} g]_- = K(f) D^{-1} g - f D^{-1} K^*(g) \quad (59)$$

Another useful technical identity involves a product of two pseudo-differential operators of the form $X_i = f_i D^{-1} g_i$, $i = 1, 2$:

$$X_1 X_2 = X_1(f_2) D^{-1} g_2 + f_1 D^{-1} X_2^*(g_1) \quad (60)$$

where $X_1(f_2) = f_1 \partial_x^{-1}(g_1 f_2)$, etc.. . From the above identity it follows the relation [18]:

$$\left(L^k\right)_- = \sum_{i=1}^m \sum_{j=0}^{k-1} L^{k-j-1}(\Phi_i) D^{-1} (L^*)^j (\Psi_i) \quad (61)$$

for the cKP Lax operator (12).

Let us also list some useful identities involving Darboux-Bäcklund -like transformation of pseudo-differential operators of the X_i -form above:

$$T_a \left(\Phi_a D^{-1} N \right) T_a^{-1} = \left(\Phi_a^2 N \right) D^{-1} \Phi_a^{-1} \quad (62)$$

$$T_a \left(M D^{-1} \Psi_a \right) T_a^{-1} = \widetilde{M} D^{-1} \left(\widetilde{L}^* (\widetilde{\Psi}_a) \right) + \left\{ T_a \left(M \partial_x^{-1} (\Psi_a \Phi_a) \right) \right\} D^{-1} \Phi_a^{-1} \quad (63)$$

$$T_a \left(M D^{-1} N \right) T_a^{-1} = \widetilde{M} D^{-1} \widetilde{N} + \left\{ T_a \left(M \partial_x^{-1} (N \Phi_a) \right) \right\} D^{-1} \Phi_a^{-1} \quad (64)$$

$$\left(\widetilde{L}^* \right)^s (\widetilde{\Psi}_a) = -\Phi_a^{-1} \partial_x^{-1} \left(\Phi_a (L^*)^{s-1} (\Psi_a) \right) \quad (65)$$

where Φ_a is one of the eigenfunctions of a cKP Lax operator L (12) and

$$\begin{aligned} T_a &\equiv \Phi_a D \Phi_a^{-1} \quad , \quad \widetilde{\Psi}_a = \Phi_a^{-1} \\ \widetilde{M} &\equiv T_a(M) = \Phi_a \partial_x \left(\Phi_a^{-1} M \right) \quad , \quad \widetilde{N} \equiv T_a^{-1*}(N) = -\Phi_a^{-1} \partial_x^{-1} (\Phi_a N) \end{aligned}$$

Finally, let us recall the following important composition formula for Wronskians [19] :

$$T_k T_{k-1} \cdots T_1(f) = \frac{W_k(f)}{W_k} \quad (66)$$

where

$$T_j = \frac{W_j}{W_{j-1}} D \frac{W_{j-1}}{W_j} = \left(D + \left(\ln \frac{W_{j-1}}{W_j} \right)' \right) \quad ; \quad W_0 = 1 \quad (67)$$

$$W_k \equiv W_k[\psi_1, \dots, \psi_k] = \det \left\| \partial_x^{i-1} \psi_j \right\| \quad , \quad W_{k-1}(f) \equiv W_k[\psi_1, \dots, \psi_{k-1}, f] \quad (68)$$

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